6. cvičení - řešení

24. 11. 2022

Příklad 1 a.

$$\lim_{n\to\infty} \frac{2^n \sqrt[3]{n^3+n^2} - n\sqrt[3]{8^n+1}}{\sqrt[n]{2^{n^2}+1}} = \lim_{n\to\infty} \frac{2^n \left(\sqrt[3]{n^3+n^2} - n\sqrt[3]{1+\frac{1}{8^n}}\right)}{2^n \sqrt[n]{1+\frac{1}{2^{n^2}}}}$$

$$= \lim_{n\to\infty} \frac{\sqrt[3]{n^3+n^2} - n\sqrt[3]{1+\frac{1}{8^n}}}{\sqrt[n]{1+\frac{1}{2^{n^2}}}} \cdot \frac{\sqrt[3]{(n^3+n^2)^2} + \sqrt[3]{n^3+n^2}\cdot n\sqrt[3]{1+\frac{1}{8^n}} + n^2\sqrt[3]{(1+\frac{1}{8^n})^2}}{\sqrt[3]{(n^3+n^2)^2} + \sqrt[3]{n^3+n^2}\cdot n\sqrt[3]{1+\frac{1}{8^n}} + n^2\sqrt[3]{(1+\frac{1}{8^n})^2}} =$$

$$= \lim_{n\to\infty} \frac{n^3+n^2-n^3+\frac{n^3}{8^n}}{\sqrt[n]{1+\frac{1}{2^{n^2}}}\cdot n^2 \cdot \left(\sqrt[3]{\left(1+\frac{1}{n}\right)^2} + \sqrt[3]{1+\frac{1}{n}}\cdot\sqrt[3]{1+\frac{1}{8^n}} + \sqrt[3]{(1+\frac{1}{8^n})^2}\right)} =$$

$$= \lim_{n\to\infty} \frac{1+\frac{n}{8^n}}{\sqrt[n]{1+\frac{1}{2^{n^2}}}\cdot \left(\sqrt[3]{\left(1+\frac{1}{n}\right)^2} + \sqrt[3]{1+\frac{1}{n}}\cdot\sqrt[3]{1+\frac{1}{8^n}} + \sqrt[3]{(1+\frac{1}{8^n})^2}\right)} \overset{\text{VoAL}}{=} \frac{1}{3}$$

Poslední rovnost jsme získali díky opakovénu použití věty o dvou strážnících na výrazy

$$c_n = \sqrt[n]{1+\frac{1}{2^{n^2}}},$$

$$d_n = \sqrt[3]{\left(1+\frac{1}{n}\right)^2} + \sqrt[3]{1+\frac{1}{n}}\cdot\sqrt[3]{1+\frac{1}{8^n}} + \sqrt[3]{(1+\frac{1}{8^n})^2}.$$

Pak máme

$$\sqrt[n]{1} \le c_n \le \sqrt[n]{2}$$

$$\lim_{n\to\infty} \sqrt[n]{1} = \lim_{n\to\infty} \sqrt[n]{2} = 1.$$

Dále máme

$$\sqrt[3]{(1)^2} + \sqrt[3]{1}\cdot\sqrt[3]{1} + \sqrt[3]{(1)^2} \le d_n \le 3\cdot\sqrt[3]{\left(1+\frac{1}{n}\right)^2}$$

$$\lim_{n\to\infty} \sqrt[3]{(1)^2} + \sqrt[3]{1}\cdot\sqrt[3]{1} + \sqrt[3]{(1)^2} = \lim_{n\to\infty} 3\cdot\sqrt[3]{\left(1+\frac{1}{n}\right)^2} = 3$$

Příklad 1 b.

$$\lim_{n\to\infty}\left(\sqrt{n^2+\sqrt{n}+1}-\sqrt{n^2+2\sqrt{n}+3}\right)\frac{\sqrt[n]{n+n^n}}{\lfloor\sqrt{n+2}\rfloor}$$
$$=\lim_{n\to\infty}\frac{n^2+\sqrt{n}+1-(n^2+2\sqrt{n}+3)}{\sqrt{n^2+\sqrt{n}+1}+\sqrt{n^2+2\sqrt{n}+3}}\frac{\sqrt[n]{n+n^n}}{\lfloor\sqrt{n+2}\rfloor}$$
$$=\lim_{n\to\infty}\frac{-\sqrt{n}-2}{\sqrt{n^2+\sqrt{n}+1}+\sqrt{n^2+2\sqrt{n}+3}}\frac{\sqrt[n]{n+n^n}}{\lfloor\sqrt{n+2}\rfloor}$$
$$=\lim_{n\to\infty}\frac{\sqrt{n}}{n}\frac{-1-\frac{2}{\sqrt{n}}}{\sqrt{1+\frac{1}{\sqrt{n}}+\frac{1}{n}}+\sqrt{1+\frac{2}{\sqrt{n}}+\frac{3}{n}}}\frac{\sqrt[n]{n+n^n}}{\lfloor\sqrt{n+2}\rfloor}$$
$$\overset{\text{AL, spoj.}}{=}-\frac{1}{2}\lim_{n\to\infty}\frac{\sqrt{n}}{n}\frac{\sqrt[n]{n+n^n}}{\lfloor\sqrt{n+2}\rfloor}\overset{\text{AL}}{=}-\frac{1}{2}\lim_{n\to\infty}\frac{\sqrt{n}}{\lfloor\sqrt{n+2}\rfloor}\lim_{n\to\infty}\frac{\sqrt[n]{n^n+n}}{n}=(*)$$

Platí

$$\sqrt{n}-1\le\lfloor\sqrt{n}\rfloor\le\lfloor\sqrt{n+2}\rfloor\le\sqrt{n+2},$$

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$$\frac{\sqrt{n}}{\sqrt{n+2}}\le\frac{\sqrt{n}}{\lfloor\sqrt{n+2}\rfloor}\le\frac{\sqrt{n}}{\sqrt{n}-1}.$$

Zároveň

$$\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+2}}=\lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{2}{n}}}\overset{\text{AL, spoj.}}{=}1$$

a

$$\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n}-1}=\lim_{n\to\infty}\frac{1}{1-\frac{1}{\sqrt{n}}}\overset{\text{AL, RŠ}}{=}1.$$

Z Věty o dvou policajtech pak platí $\lim_{n\to\infty}\frac{\sqrt{n}}{\lfloor\sqrt{n+2}\rfloor}=1$. Dále platí

$$1=\frac{\sqrt[n]{n^n}}{n}\le\frac{\sqrt[n]{n^n+n}}{n}\le\frac{\sqrt[n]{2n^n}}{n}=\sqrt[n]{2}$$

a

$$\lim_{n\to\infty}1=1=\lim_{n\to\infty}\sqrt[n]{2}.$$

Opět použitím Věty o dvou policajtech dostáváme

$$(*)=-\frac{1}{2}\cdot1\cdot1=-\frac{1}{2}$$

Příklad 1 c.

$$\lim_{n\to\infty}\left(\sqrt[3]{2n+a}-\sqrt[3]{2n+b}\right)\cdot\sqrt[3]{(n+1)(3n+2)} =$$

$$= \lim_{n\to\infty}\sqrt[3]{(n+1)(3n+2)}\left(\sqrt[3]{2n+a}-\sqrt[3]{2n+b}\right)\cdot\frac{(\sqrt[3]{2n+a})^2+\sqrt[3]{2n+a}\cdot\sqrt[3]{2n+b}+(\sqrt[3]{2n+b})^2}{(\sqrt[3]{2n+a})^2+\sqrt[3]{2n+a}\cdot\sqrt[3]{2n+b}+(\sqrt[3]{2n+b})^2} =$$

$$= \lim_{n\to\infty}\frac{((2n+a)-(2n+b))\cdot\sqrt[3]{(n+1)(3n+2)}}{(\sqrt[3]{2n+a})^2+\sqrt[3]{2n+a}\cdot\sqrt[3]{2n+b}+(\sqrt[3]{2n+b})^2} =$$

$$= \lim_{n\to\infty}\frac{(a-b)n^{\frac{2}{3}}\sqrt[3]{(1+\frac{1}{n})(3+\frac{2}{n})}}{n^{\frac{2}{3}}\left((\sqrt[3]{2+\frac{a}{n}})^2+\sqrt[3]{2+\frac{a}{n}}\cdot\sqrt[3]{2+\frac{b}{n}}+(\sqrt[3]{2+\frac{b}{n}})^2\right)} \overset{\text{VoAL}}{=} \frac{(a-b)\cdot\sqrt[3]{3}}{3\cdot 2^{\frac{2}{3}}} = (a-b)\cdot 6^{-\frac{2}{3}}$$

Použili jsme vzorec
$A^n - B^n = (A-B)(A^{n-1}+A^{n-2}B+A^{n-3}B^2+\cdots+A^2B^{n-3}+AB^{n-2}+B^{n-1})$.

Příklad 1 d. Pro $k\in\mathbb{N}$ označme $P_{\le k}(n)$ polynom proměnné n stupně nejvýše k.

$$\lim_{n\to\infty}\frac{(n^2+1)^{100}-(n+2)^{200}+400n^{199}}{1+2+\ldots+n^{99}}$$

$$= \lim_{n\to\infty}\frac{n^{200}+100n^{198}+P_{\le 196}(n)-n^{200}-400n^{199}-4\cdot\binom{200}{2}n^{198}-P_{\le 197}(n)+400n^{198}}{\frac{n^{99}(n^{99}+1)}{2}}$$

$$= \lim_{n\to\infty}\frac{n^{198}(100-79600+\frac{P_{\le 197}(n)}{n^{198}})}{n^{198}(\frac{1}{2}+\frac{1}{n^{99}})} = \lim_{n\to\infty}\frac{100-79600+\frac{P_{\le 197}(n)}{n^{198}}}{\frac{1}{2}+\frac{1}{n^{99}}} \overset{\text{AL}}{=} -159000.$$

Příklad 2

Vzorové řešení doc. Johanise.